PUTNAM PRACTICE SET 1

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Problem 1. Let m be a positive integer, let a be a positive real number and let θ be a real number. Prove there exist m quadratic polynomials $Q_1(x), \dots, Q_m(x) \in \mathbb{R}[x]$ such that

$$x^{2m} - 2a^m x^m \cos(m\theta) + a^{2m} = Q_1(x) \cdot Q_2(x) \cdots Q_m(x).$$

Solution. We could conclude our solution easily by noting that once a root ξ exists for our polynomial with real coefficients, then also its complex conjugate $\bar{\xi}$ is also a root and therefore, we could group all the complex (non-real) solutions of our polynomial in pairs which leads to quadratics, while the remaining real roots we arbitrarily group them in pairs, which also leads to quadratic polynomials and therefore, we obtain m quadratic polynomials. However, this does not give a complete answer about the actual coefficients of the quadratic polynomials Q_i; we can give such a complet answer as follows.

We have that $\cos(m\theta) = \frac{e^{im\theta}+e^{-im\theta}}{2}$ and so, letting $z_0 := e^{i\theta}$, we have that

$$P(x) := x^{2m} - 2a^m x^m \cos(m\theta) + a^{2m} = x^{2m} - a^m x^m \left(z_0^m + z_0^{-m}\right) + a^{2m}$$

and so,

$$P(x) = \left(x^m - a^m z_0^m\right) \cdot \left(x^m - a^m z_0^{-m}\right).$$

Therefore, letting $\zeta_1, \dots, \zeta_m$ be all the distinct m-th roots of unity, then we have (note that the set of m-th roots of unity is closed under complex conjugation and also, note that $\bar{z_0} = z_0^{-1}$)

$$P(x)$$

$$= \prod_{j=1}^{m} \left(x - \zeta_j a z_0\right) \cdot \prod_{j=1}^{m} \left(x - \bar{\zeta_j} a z_0^{-1}\right)$$

$$= \prod_{j=1}^{m} \left(x^2 - 2a\mathrm{Re}(\zeta_j z_0)x + a^2\right).$$

Therefore, the desired polynomials with real coefficients are $Q_j(x) := x^2 - 2a\mathrm{Re}(\zeta_j z_0)x + a^2$.

Problem 2. For a matrix, the following operations are considered acceptable:

- change the sign of each entry on any given row;
- change the sign of each entry on any given column;
- switch two rows;
- switch two columns.

Prove that we cannot use the above operations to transform the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

into the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}.$$

Solution. The two determinants are only changed in their sign by the given operations. On the other hand, the first determinant is nonzero, while the second determinant equals 0.

Problem 3. Find the infinimum of $a^2 + b^2$ over all the possible pairs (a, b) of real numbers with the property that the equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$

has 4 distinct real roots.

Solution. Letting $f(x) := x^4 + ax^3 + bx^2 + ax + 1$, we note that $x^4 \cdot f\left(\frac{1}{x}\right) = f(x)$, thus proving that the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of $f(x)$ have the property that grouping them in 2 pairs, they are of the form $\left(\beta, \frac{1}{\beta}\right)$. So, we write them as $\beta_1, \frac{1}{\beta_1}, \beta_2, \frac{1}{\beta_2}$ and then write $f(x)$ as the product of two quadratic polynomials

$$Q_j(x) := (x - \beta_j) \cdot \left(x - \frac{1}{\beta_j}\right) \text{ for } j = 1, 2.$$

So, $f(x) = (x^2 + \gamma_1 x + 1) \cdot (x^2 + \gamma_2 x + 1)$ for some distinct real numbers γ_1 and γ_2 (note that they are distinct since $\beta_1 \neq \beta_2$ because the roots α_i are all distinct). Furthermore, because the β_j's are real numbers, we have that the γ_j's must be real numbers satisfying the inequality $|\gamma_j| > 2$ (note that the inequality is strict because the α_i's are distinct.) Now,

$$x^4 + ax^3 + bx^2 + ax + 1 = (x^2 + \gamma_1 x + 1)(x^2 + \gamma_2 x + 1)$$

yields $a = \gamma_1 + \gamma_2$ and $b = \gamma_1\gamma_2 + 2$. So,

$$a^2 + b^2 = (\gamma_1^2 + \gamma_2^2) + (6\gamma_1\gamma_2 + \gamma_1^2\gamma_2^2 + 4).$$

Clearly, for any fixed real number $\zeta := \gamma_1\gamma_2$, the smallest quantity for $a^2 + b^2$ is attained when $\gamma_1 = \gamma_2 = \zeta^{1/2}$ if $\zeta \geq 0$, while if $\zeta < 0$, then the minimum is attained when $\gamma_1 = |\zeta|^{1/2} = -\gamma_2$ (or conversely). Furthermore, $a^2 + b^2$ is minimized when $\zeta < 0$ and so, we have that

$$a^2 + b^2 \geq 4\zeta + \zeta^2 + 4 = (\zeta + 2)^2 > (-4 + 2)^2 = 4$$

because $\zeta = \gamma_1\gamma_2$ and we know (based on our assumption that $\zeta < 0$) that $\gamma_1 > 2$ while $\gamma_2 < -2$ (or vice-versa). So, the infimum of $a^2 + b^2$ equals 4 because we can choose γ_1 arbitrarily close to 2 and choose γ_2 arbitrarily close to -2.

Problem 4. Let k be a positive integer. Find the set of all tuples $(a_1, \dots, a_{k+1})$ of non-negative integers satisfying the following properties:

- $a_1 = 0$.
- $|a_i - a_{i+1}| = 1$ for $i = 1, \dots, k$.

Solution. Let M be the set of all these sequences. We let N be the set of all sequences of integers $\{a_i\}_{1\le i\le k+1}$ satisfying the properties:

- $a_1 = 0$ and $a_{k+1} \ge 0$.
- $|a_i - a_{i+1}| = 1$ for $i = 1, \dots, k$.

In other words, we do not impose the condition that $a_i \ge 0$ for $i = 2, \dots, k$ for the sequences in N; so, in particular, $M \subset N$.

Now, we consider a sequence $\{a_i\}_{1\le i\le k+1}$ which lies in $N \setminus M$. Then there exists a smallest integer i such that $a_i < 0$. From the definition of the sequence, we must have that $a_i = -1$ (since the difference between consecutive elements of the sequence is 1, in absolute value). We consider the following function $\varphi : N \setminus M \longrightarrow N$ given by

$$\varphi\left(\{a_j\}_{1\le j\le k+1}\right) = \{b_j\}_{1\le j\le k+1}, \text{ where}$$

$$b_j = -a_j \text{ for } 1 \le j \le i \text{ and } b_j = a_j + 2 \text{ for } i+1 \le j \le k+1.$$

Letting $P \subset N$ be the set containing all sequences from N with the additional property that $a_{k+1} \ge 2$, we claim the following.

Claim. The function φ restricts to a bijection between $N \setminus M$ and P.

Proof of Claim. First we see that φ is injective. We see from the definition of φ that if two sequences $\{a_j\}_{1\le j\le k+1}$ and $\{a'_j\}_{1\le j\le k+1}$ are mapped into the same sequence $\{b_j\}_{1\le j\le k+1}$, then there exists a unique least integer i with the property that $b_i = 1$ and therefore, for both sequences, we have the same index i as being the least integer for which $a_i < 0$, respectively $a'_i < 0$. Then the definition of φ yields that $a_j = a'_j$ for $1 \le j \le i$ and also for $i+1 \le j \le k+1$. Thus, φ is injective.

Next, we see that indeed the image $\varphi(N \setminus M)$ is contained in P because $b_{k+1} = a_{k+1} + 2$ and $a_{k+1} \ge 0$.

Now, conversely: given any sequence $\{b_j\}_{1\le j\le k+1} \in P$, we simply consider the first integer i such that $b_i = 1$; note that such an integer must exist since $b_{k+1} \ge 2$. Then, we define the sequence $\{a_j\}_{1\le j\le k+1} \in N \setminus M$, as follows:

$$a_j = -b_j \text{ for } 1 \le j \le i \text{ and } a_j = b_j - 2 \text{ for } i+1 \le j \le k+1.$$

We immediately see that $\varphi\left(\{a_j\}_{1\le j\le k+1}\right) = \{b_j\}_{1\le j\le k+1}$, thus showing that φ is also surjective, which finishes the proof of our Claim.

The Claim above yields that $|N \setminus M| = |P|$ and so,

$$|M| = |N| - |N \setminus M| = |N| - |P| = |E|,$$

where $E \subset N$ is the set of sequences $\{a_i\}_{1\le i\le k+1}$ which satisfy the following properties:

- $a_1 = 0$.
- $|a_i - a_{i+1}| = 1$ for $1 \le i \le k$.
- $a_{k+1} \in \{0, 1\}$.

Now, letting $\epsilon_i := a_{i+1} - a_i$ for $1 \le i \le k$, we see that the set E is in bijection to the set of all k-tuples $(\epsilon_1, \dots, \epsilon_k) \in \{-1, 1\}^k$ with the property that $\sum_{i=1}^k \epsilon_i \in \{0, 1\}$. In other words, we know that in the set of k-tuples $(\epsilon_1, \dots, \epsilon_k)$ we have precisely

$[k/2]$ numbers equal to -1, while the other numbers equal 1. So, the cardinality of P (and therefore, the desired cardinality of M) equals $\binom{k}{[k/2]}$.

Problem 5. Prove that for each positive integer n, we have

$$2^n \cdot \prod_{k=1}^{n} \sin\left(\frac{k\pi}{2n+1}\right) = \sqrt{2n+1}.$$

Solution. We note that for each integer $n \geq 0$, there exists a polynomial $Q_n(x) \in \mathbb{R}[x]$ of degree $2n+1$ such that $\sin\left((2n+1)x\right) = Q_n\left(\sin(x)\right)$; for example, $Q_0(x) = x$, while $Q_1(x) = 3x - 4x^3$. Furthermore, we know that the roots of $Q_n(x)$ are $\sin\left(\frac{2k\pi}{2n+1}\right)$ for $k = 0, \dots, 2n$. So, we can write $Q_n(x) = x \cdot P_n(x)$, where $P_n(x)$ has the roots $\pm\sin\left(\frac{2k\pi}{2n+1}\right)$ for $k = 1, \dots, n$; furthermore, these roots of $P(x)$ can be written as $\pm\sin\left(\frac{k\pi}{2n+1}\right)$. Thus, letting a_n be the leading coefficient of $P_n(x)$ and also letting a_0 be the constant term of $P_n(x)$, we get that the product

$$p_n := \prod_{k=1}^{n} \sin\left(\frac{k\pi}{2n+1}\right)$$

satisfies $(-1)^n \cdot p_n^2 = \frac{a_0}{a_n}$. So, we need to compute a_0 and a_n since then the formula would follow (note that we know that $p_n > 0$).

We claim that $a_n = (-1)^n \cdot 4^n$, while $a_0 = 2n+1$. First, we note that $a_0 = P_n(0)$. Now, we note that $\cos((2n+1)x) = \cos(x) \cdot R_n(\sin(x))$, where $R_n(x) \in \mathbb{R}[x]$ is a polynomial of degree $2n$. Then we have

$$Q_n(x) = Q_{n-1}(x) \cdot (1 - 2x^2) + 2x(1 - x^2) \cdot R_{n-1}(x)$$

and so,

$$P_n(x) = P_{n-1}(x) \cdot (1 - 2x^2) + 2(1 - x^2) \cdot R_{n-1}(x). \tag{1}$$

Hence $P_n(0) = P_{n-1}(0) + 2R_{n-1}(0)$. On the other hand, we know that $P_0(0) = 1$ (since $P_0(x) = 1$) and we prove that $R_n(0) = 1$ for all n. This follows because

$$R_n(x) = R_{n-1}(x) \cdot (1 - 2x^2) - 2x^2 Q_{n-1}(x), \tag{2}$$

and so, $R_n(0) = R_{n-1}(0)$ for all $n \geq 1$. So, we're left to prove that $a_n = (-4)^n$. Again, this follows from (1) by combining it with (2) because the first equation yields

$$a_n = a_{n-1} \cdot (-2) - 2b_{n-1},$$

where b_{n-1} is the leading coefficient of $R_{n-1}(x)$, while the second equation yields

$$b_n = b_{n-1} \cdot (-2) - 2a_{n-1}.$$

We start with $a_0 = 1$ and $a_1 = -4$, while $b_0 = 1$ and $b_1 = -4$ and so, inductively, we derive that $a_n = b_n = (-4)^n$, as desired.

Problem 6. Let P be a set of 5 distinct prime numbers and let B be the set of all 15 numbers obtained as a product of two numbers from P, not necessarily distinct. We partition B into 5 disjoint sets $C_1, \cdots, C_5$, each one of them containing precisely 3 elements from B, and moreover having the property that for each $i = 1, \dots, 5$, there is a prime dividing each of the 3 numbers from the set C_i. How many possible partitions of B into 5 such subsets are there?

Solution. Each C_i corresponds to some prime $p_{j(i)}$ such that each element of C_i is divisible by $p_{j(i)}$; furthemore, $j(1), \cdots, j(5)$ is a permutation of $\{1, \dots, 5\}$ since if the same prime p_j divides the elements of two sets C_{i_1} and C_{i_2}, then it would mean there exist 6 elements of B divisible by p_j, contradiction. So, we may assume (since we deal with a partition), that each element of C_i is divisible by p_i for $i = 1, \dots, 5$. Hence, $p_i^2 \in C_i$ for each $i = 1, \dots, 5$. There are then 6 possibilities for the other two elements of C_1; by symmetry, we assume

$$C_1 = \{p_1^2, p_1p_2, p_1p_3\}.$$

We know that $p_2p_3 \in C_2$ or $p_2p_3 \in C_3$; again, by symmetry, we assume $p_2p_3 \in C_2$. Then there are 2 options for the remaining element of C_2; let's assume $C_2 = \{p_2^2, p_2p_3, p_2p_4\}$. We have so far $6 \cdot 2 \cdot 2 = 24$ options and we'll prove that with our choices so far, C_3, C_4 and C_5 are prescribed. Indeed, $p_3^2 \in C_3$ and we know that $p_1p_3, p_2p_3 \notin C_3$, which means that $C_3 = \{p_3^2, p_3p_4, p_3p_5\}$. Then $p_4^2 \in C_4$ but $p_2p_4, p_3p_4 \notin C_4$; so, $C_4 = \{p_p^2, p_4p_1, p_4p_5\}$. Finally, $p_5^2 \in C_5$ but $p_3p_5, p_4p_5 \notin C_5$, and therefore $C_5 = \{p_5^2, p_1p_5, p_2p_5\}$. So, indeed there are 24 possible partitions (note that two partitions are not counted as being different if they contain the exact same 5 sets C_i but listed in a different order).